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Closed form solution for a nonlocal elastic bar in tension

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Abstract

A simple mechanical one-dimensional problem in the context of nonlocal (integral) elasticity is solved analytically. The nonlocal elastic material behaviour is described by the "Eringen model" whose nonlocality features all reside in the constitutive relation. This relation, of integral type, contains an attenuation function (usually assumed symmetric) aimed to capture the diffusion process of the nonlocality effects; it also exhibits a convolution format. The governing equation is a Fredholm integral equation of second kind whose analytical treatment, even for the usual choice of a symmetric kernel, is not easy to develop. In the present paper, assuming a specific shape for the attenuation function, a *closed form solution* in terms of strains is alternatively obtained by solving a Volterra integral equation of second kind. The latter can be easily solved with standard techniques, at least for the adopted kernel, taking also advantage from the symmetry of the solution. Such a closed form solution is an essential result to validate the effectiveness of numerical procedures aimed to solve more complex mechanical problems in the context of nonlocal elasticity.

Keywords: Nonlocal elasticity; Diffusion process; Attenuation function; Volterra integral equation; Exact solution

1. Introduction

The early ideas of nonlocal elasticity can be traced back to the pioneristic works of Kröner (1967), Kunin (1967), Krumhansl (1968) and Edelen (1969). Improved formulations were then presented in Edelen and Laws (1971) and Eringen (1972a, 1972b, 1976), till the quite recent contributions of Rogula (1982), Eringen (1987), etc. The motivation of such studies grounds on the fact that one of the main drawbacks of classical (local) elasticity theories is that the latter are not able to handle elastic problems in the presence of sharp geometrical singularities, for which they in fact lead to incongruities. A striking example is, typically, in the context of continuum fracture mechanics, the singular stress field predicted at a sharp crack-tip. Such *inability* of the local theory is indeed to be interpreted as the attainment of the limit of applicability of the theory itself which is reached when some internal characteristic length and/or time scale (such as atomic

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distance, granular distance, relaxation time, lattice parameter, etc.) become comparable with the external scales (such as wave length, period, area of application of loads, etc.). In other words, the local continuum elasticity theory is sufficient for the description of physical phenomena adequately represented by the behaviour of a very large number of molecules gathered together, otherwise the atomic lattice theory is necessary (Eringen, 1987). Nevertheless, all the molecular or atomic theories, when applied to solve problems of practical interest, give rise to an huge amount of cumbersome computations whose results, when reachable, have to be compared with the ones furnished by experimental investigations. The latter are usually referred to statistical averages of phenomena arising at atomic level, that is again referred to an aggregate of molecules or, in a certain sense, to a continuum.

A possible solution to outlined difficulties is offered by a continuum approach endowed with information regarding the behaviour of the *material microstructure*, i.e. applied to the continuum grounding on the concept of neighbouring points linked together by *long range forces* or, equivalently, on the capacity exhibited—at a microstructural level—by an elastic material to transmit information to neighbouring points within a certain distance. This distance, known as *internal length material scale* is an essential parameter to be introduced in the theory in order to take into account phenomena arising at the microstructure. In this view, the denied inability of the local continuum elasticity theory arises from the circumstance that such theory contains no information about the long range forces. The internal length material scale is absent.

In nonlocal continuum theories, the internal length enters the constitutive equations simply as a material parameter allowing to work with *nonlocal variables* conceived, for example, as weighted average of local variables over all the material points in the body. The internal length parameter drives the weighting process performed over a variable at a certain point. As highlighted in Rogula (1982), (see also the recent contribution of Polizzotto (2001)), a distinction could be made at this point on the variety of approaches or techniques belonging to nonlocal elasticity. Namely, the nonlocal elasticity approach can be qualified as "integral" or "strongly nonlocal" when it expresses the stress at a point of a material domain as a weighted value of the entire strain field. It can be qualified as "gradient" or "weakly nonlocal" when the stress is expressed as a function of the strain and its gradients at the same points. Alternatively, an internal length scale in a continuum can also be introduced by considering *polar elasticity* as in Cosserat theory (Sluys, 1992). After all, besides any possible and surely effective classification, when dealing with a nonlocal approach conceiving, for example, the existence of *body couples* or making use of a *gradient operator* in the constitutive law or, moreover, of an *integral operator*, the notion of distance, or surface or volume is involved respectively and at last the idea of an internal length is invoked. The above reasonings should deserve further and deeper investigations but they are out of the aim of the present paper. The authors' attention was in fact focused on the search of an *exact* or *closed form solution* for a practical, even though very simple, mechanical problem and this to provide a reference solution for numerical approaches, first proposed in Polizzotto (2001), object of an ongoing research (Polizzotto et al., 2001).

In this paper the so-called Eringen model of nonlocal integral elasticity is considered. The model, presented by Eringen and co-workers (Eringen and Kim, 1974; Eringen et al., 1977; Eringen, 1978, 1979) with reference to linear homogeneous isotropic continua is characterized by a nonlocal elasticity theory applied in a simplified fashion, in the sense that it differs from the classical local one only for the stress–strain constitutive relation which exhibits a convolution format. The model, briefly summarized in Section 2, is applied to a simple 1-D problem, i.e. a nonlocal bar of finite length L, constant cross-section and subjected to (uniform) tension. To the problem position is devoted Section 3.1 where some remarks on the consistent enforcement of the boundary conditions are also given. The development of a closed form solution for the mentioned bar problem and a specific choice of the attenuation function is addressed in Sections 3.2,3. Diagrams of the analytical solutions for different values of material parameters are plotted in Section 3.4. Concluding remarks are finally drawn in Section 4.

2. The Eringen model

The model proposed by Eringen and co-workers (Eringen and Kim, 1974; Eringen et al., 1977) is based on the key idea that the long-range forces, responsible of the nonlocal behaviour of a homogeneous isotropic elastic material, are adequately described by a constitutive relation of the form:

$$\boldsymbol{\sigma}(\mathbf{x}) = \int_V K(\mathbf{x}, \mathbf{x}')\mathbf{D} : \boldsymbol{\varepsilon}(\mathbf{x}')\,\mathrm{d}\mathbf{V}' \quad \forall\ \mathbf{x} \in \mathbf{V}, \tag{1}$$

where: V is the volume of the 3D domain referred to a Cartesian orthogonal co-ordinate system $\mathbf{x} = (x_1, x_2, x_3)$ and filled by the continuum conceived as an aggregate of material particles, linked one another by cohesive bonds (between adjacent particles) and long range forces or legaments (between nonadjacent particles); $\mathrm{d}V' := \mathrm{d}V(\mathbf{x}')$; $\boldsymbol{\sigma}(\mathbf{x})$ and $\boldsymbol{\varepsilon}(\mathbf{x})$ are the second order tensors representing stress and strain fields at $\mathbf{x}$, respectively; $\mathbf{D}$ denotes the elastic moduli fourth-rank tensor of classical (local) isotropic elasticity. Finally the scalar function $K(\mathbf{x}, \mathbf{x}')$ is the *attenuation* or *influence function* aimed to *inject* in the constitutive law the nonlocality effects at the field points $(\mathbf{x})$ produced by the (local) strain at the source $(\mathbf{x}')$.

Typically $K(\mathbf{x}, \mathbf{x}')$ is a function of the Euclidean distance $|\mathbf{x}' - \mathbf{x}|$, it is positive and decays more or less rapidly with increasing distance. To this concern Polizzotto (2001) proposes the use of the *geodetical distance*, as the "path of minimum length not intersecting the boundary surface of the body", and this to correctly describe the diffusion process of the nonlocality effects even when cracks, holes or, in general, nonconvex domains are considered. Hereafter reference is made only to convex domains without holes or inside openings so that the geodetical distance and the Euclidean one are obviously coincident.

The role played by the distance between the source point $(\mathbf{x}')$, at which a local variable "acts", and the point $(\mathbf{x})$, where its nonlocal effects are detected, is crucial in the description of the diffusion process of the nonlocality effects captured by the postulated constitutive law. The distance between the two points is meaningful only if compared with a material parameter, namely the internal length scale ℓ. The mentioned distance can then be qualified as "large" or "small" only relatively to ℓ. At small distances, the nonlocality effects propagate almost unaltered, whereas they reduce sensibly at large distances. Finally, beyond the maximum distance from the source point within which the diffusion process is physically meaningful—namely the *influence distance*—the nonlocality effects are almost vanishing. The influence distance is a multiple of the internal length ℓ and both have to be considered much smaller than the smallest dimension of the body or probe specimen. The attenuation function has then to be in effect a scalar function of the ratio $|\mathbf{x}' - \mathbf{x}|/\ell$ and this is the usual choice in the literature.

An additional comment on another essential feature of the attenuation function is related to the circumstance that if $\ell \to 0$, i.e. in the limit of a local elastic material behaviour, Eq. (1) has to transform in the classical local elastic relation: $\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{D} : \boldsymbol{\varepsilon}(\mathbf{x})$. This essential requisite can be accomplished by requiring $K(|\mathbf{x}' - \mathbf{x}|/\ell)$ to become a Dirac delta for $\ell \to 0$, i.e. imposing the condition:

$$\int_{V_\infty} K(|\mathbf{x}' - \mathbf{x}|/\ell)\,\mathrm{d}V' = 1, \tag{2}$$

in which V_∞ is the infinite domain embedding V, if V is finite.

It is worth noting that Eq. (1) can be alternatively posed in the form:

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{D} : \widehat{\boldsymbol{\varepsilon}}(\mathbf{x}), \tag{3}$$

in which the classical Hooke law format is recovered through the introduction of the *nonlocal strain field* $\widehat{\boldsymbol{\varepsilon}}(\mathbf{x})$ defined as:

$$\widehat{\boldsymbol{\varepsilon}}(\mathbf{x}) = \int_V K(\mathbf{x},\mathbf{x}')\boldsymbol{\varepsilon}(\mathbf{x}')\,\mathrm{d}V'. \tag{4}$$

According to Eqs. (3) and (4) the long range forces, which arise in a homogeneous isotropic elastic material due to a small strain field $\boldsymbol{\varepsilon}$, are described by the stress field $\boldsymbol{\sigma}$ related to a nonlocal strain field $\widehat{\boldsymbol{\varepsilon}}$ through the relevant elastic moduli tensor $\mathbf{D}$ of isotropic local elasticity. The nonlocal strain field $\widehat{\boldsymbol{\varepsilon}}$ is connected to the local one, $\boldsymbol{\varepsilon}$, by the convolution formula (4).

Finally, with concern to the constitutive equation of the Eringen model expressed by Eq. (1), a different shape of it can be set on conceiving the nonlocal elastic material as a two-phase elastic material. Precisely, phase 1 material (of volume fraction ξ_1) complying with local elasticity and phase 2 material (of volume fraction ξ_2) complying with nonlocal elasticity. The constitutive relation can then be given in the shape (see e.g. Eringen, 1987; Altan, 1989):

$$\boldsymbol{\sigma}(\mathbf{x}) = \xi_1\mathbf{D} : \boldsymbol{\varepsilon}(\mathbf{x}) + \xi_2 \int_V K(\mathbf{x},\mathbf{x}')\mathbf{D} : \boldsymbol{\varepsilon}(\mathbf{x}')\,\mathrm{d}V', \tag{5}$$

with ξ_1 and ξ_2 being positive material constants and assuming $\xi_1 + \xi_2 = 1$. This general form will be considered in the following.

3. A nonlocal bar in tension

3.1. Problem position and boundary conditions

The bar of uniform cross-section A and finite length L sketched in Fig. 1(a) is considered; for simplicity, it is set $A = 1$. By hypothesis, the bar is subjected to boundary forces $F = A\bar{\sigma}$ applied at the end sections so that a uniform tensile stress $\bar{\sigma}$ is induced in the bar. The above boundary forces are, as usual in the literature, applied at the end sections, just like in a "local elasticity approach". The bar is made of a nonlocal homogeneous isotropic linear elastic material whose constitutive behaviour complies with the Eringen model given in Section 2. The apposite constitutive relation (refer to Eq. (5)) reads:

$$\sigma(x) = E\left[\xi_1\varepsilon(x) + \xi_2 \int_0^L K(x,x')\varepsilon(x')\,\mathrm{d}x'\right], \tag{6}$$

where E is the Young's modulus and $K(x,x')$ is the attenuation function centered at x, herein assumed symmetric. Since $\sigma = \bar{\sigma}$ in all cross sections, by equilibrium, Eq. (6) can be written in the equivalent form:

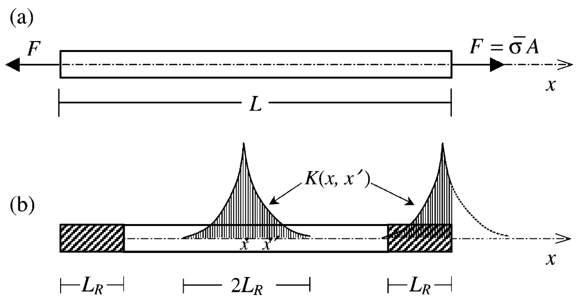

Fig. 1. Nonlocal bar in tension: (a) bar specimen of length L and applied boundary forces; (b) schematic representation of the attenuation function $K(x,x')$ and of the bar end portions, L_R denoting the *influence distance* beyond which the diffusion process is practically vanishing.

$$\varepsilon(x) = \frac{\bar{\varepsilon}}{\xi_1} - \frac{\xi_2}{\xi_1} \int_0^L K(x, x')\varepsilon(x')\,\mathrm{d}x', \tag{7}$$

where $\bar{\varepsilon} := \bar{\sigma}/E =$ uniform nonlocal strain (see Eq. (3)). The latter equation, which is a Fredholm equation of second kind, makes evident that—contrary to local elasticity—in a finite length nonlocal bar a uniform stress state is not accompanied by a uniform strain. To this concern the following remarks can be drawn.

Remark 1. The integral term of Eq. (7) is to be interpreted as the result of a *weighting process* carried on, within an *influence zone* centered at x and of length equal to twice the influence distance $L_R \ll L$, in which the attenuation function $K(x, x')$ plays the role of collecting the nonlocality effects produced at all source points x' (refer to Fig. 1(b)). Outside the above influence zone, the attenuation function is practically vanishing so as the weighting process. Grounding on condition (2) and the above observations the following condition holds:

$$\int_{-L_R}^{L_R} K(x, x')\,\mathrm{d}x' \cong \int_{-\infty}^{\infty} K(x, x')\,\mathrm{d}x' = 1. \tag{8}$$

Remark 2. At points x sufficiently far from the end sections, namely $\forall x : L_R \leqslant x \leqslant L - L_R$, the influence zone is always inside the bar specimen and thus it collects the strain sources all around x. At these points, a constant strain solution, say $\varepsilon = \bar{\varepsilon}$, is achievable by (8).

Remark 3. At points x near to the end sections, namely belonging to the *end portions* of the bar having each length L_R, the influence zone exceeds the end section (domain boundary) and it collects obviously only the strain sources at points x' belonging to the bar specimen. At these points x a constant strain solution as $\varepsilon = \bar{\varepsilon}$ is not achievable. For the posed problem the strain values are characterised by an increasing trend in the end portions and this, in fact, is the typical pattern found in the literature.

From a physical point of view, the existence of an impeding boundary, within which the strain source distribution and the consequent diffusion processes of the nonlocality effects are confined, produces some *boundary effects* that, for a clear understanding should be, or might be, treated with suitable micromechanics investigations. Furthermore, these boundary effects can be accounted for by a more appropriate definition of the boundary conditions. A recent contribution to this concern has been given in Polizzotto et al. (2001) where two different possible approaches, able to catch—at a macroscopic level—the mentioned boundary effects, have been considered. As observed in the quoted paper many intermediate interpretations are also possible and, to the authors' knowledge, a consistent assignment of the boundary conditions in a nonlocal elastic problem is an open research issue. The latter is however beyond the scope of the present study whose main goal is the determination of an *exact solution* to the nonlocal bar problem governed by Eq. (7) and this for a specific choice of the influence function and with the boundary conditions being applied in a "local fashion". Such an exact solution is definitely an essential starting point also in the direction of a more appropriate definition of the *nonlocal* boundary conditions which is actually the object of an ongoing research of the authors.

3.2. Search for the unknown strain field

As previously outlined the strain field arising in the bar is given by Eq. (7) which is a Fredholm integral equation of second kind that can, in general, be solved by application of a variety of classical techniques all grounding on series expansion of the kernel (refer e.g. to Tricomi, 1987; Krasnov et al., 1983). However, taking into account that for the mechanical problem under study the unknown strain field $\varepsilon(x)$ is symmetric with respect to the mid section $x = L/2$, Eq. (7) is split into two Volterra integral equations of the second kind whose solution is more easily obtainable.

To this aim Eq. (7) is firstly expressed in the following general shape:

$$\varepsilon(x) = C + \lambda \int_0^L K(|x - x'|)\varepsilon(x')\,\mathrm{d}x', \tag{9}$$

where $C = \bar{\varepsilon}/\xi_1 + C_0$ and $\lambda = -\xi_2\lambda_0/\xi_1$ with C_0 and λ_0 constants to be specified and the kernel is assumed as a function of the distance between the field (x) and the source (x') points. Referring to Fig. 2 the unknown strain function $\varepsilon(x)$ can be conceived as the sum of two contributions defined in the subdomains $[0, x]$ and $[x, L]$ respectively. Precisely:

$$\varepsilon(x) = \varepsilon^-(x) + \varepsilon^+(x), \tag{10}$$

with

$$\varepsilon^-(x) = H(x - x')\varepsilon(x); \quad \varepsilon^+(x) = H(x' - x)\varepsilon(x). \tag{11a, b}$$

The symbol $H(x)$ denotes the Heaviside operator defined as $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x \geqslant 0$. On combining Eqs. (9) and (10) the following equality holds:

$$\varepsilon(x) = C + \lambda \int_0^L K(|x - x'|)H(x - x')\varepsilon(x')\,\mathrm{d}x' + \lambda \int_0^L K(|x - x'|)H(x' - x)\varepsilon(x')\,\mathrm{d}x'. \tag{12}$$

On setting:

$$K^-(x - x') := K(|x - x'|)H(x - x'), \tag{13a}$$

$$K^+(x' - x) := K(|x - x'|)H(x' - x), \tag{13b}$$

viewed as *modified kernels* defined over the triangular regions depicted in Fig. 3, Eq. (12) can be rewritten as:

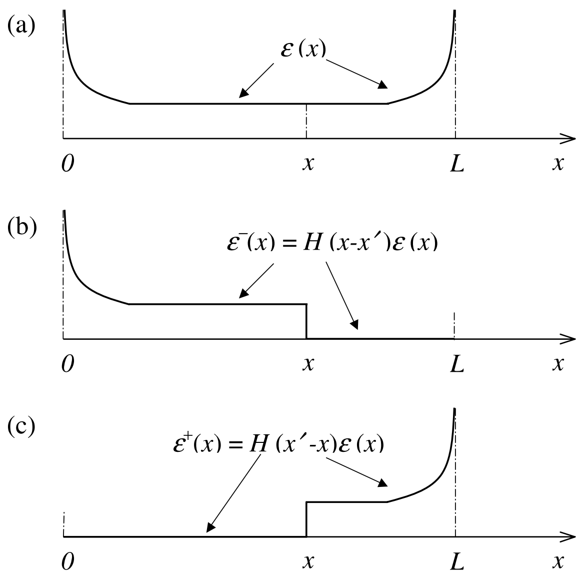

Fig. 2. Unknown strain function: (a) qualitative pattern in the entire domain $[0, L]$; (b) and (c) strain function decomposition.

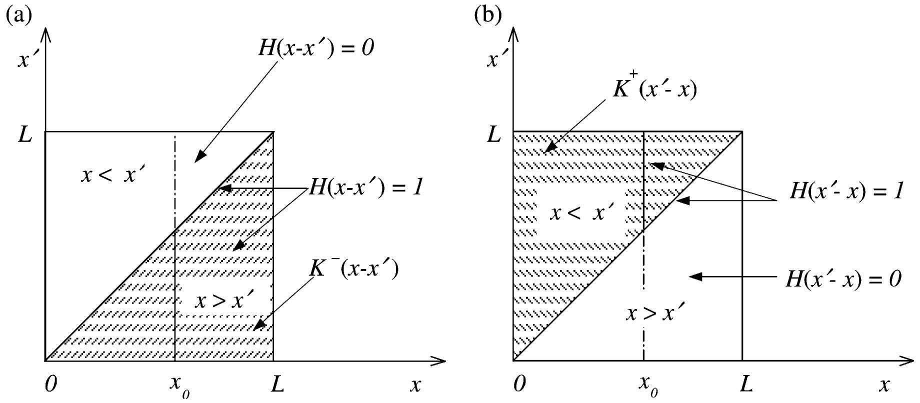

Fig. 3. Triangular regions defining the modified kernels' domains and the integration paths of Eq. (14) for fixed x (e.g. $x = x_0$).

$$\varepsilon(x) = C + \lambda \int_0^x K^-(x - x')\varepsilon^-(x')\,\mathrm{d}x' + \lambda \int_x^L K^+(x' - x)\varepsilon^+(x')\,\mathrm{d}x'. \tag{14}$$

On taking into account Eq. (10) and observing that all the quantities $(\cdot)^-$ are meaningful for $x' \leqslant x$ and the ones $(\cdot)^+$ for $x' \geqslant x$ Eq. (14) can be split as follows:

$$\varepsilon^-(x) = C^- + \lambda \int_0^x K^-(x - x')\varepsilon^-(x')\,\mathrm{d}x' \tag{15a}$$

$$\varepsilon^+(x) = C^+ + \lambda \int_x^L K^+(x' - x)\varepsilon^+(x')\,\mathrm{d}x', \tag{15b}$$

where $C = C^- + C^+$. Eqs. (15a) and (15b) are the two Volterra integral equations of the second kind in which Eq. (9) has been split up.

It is worth noting that Eq. (15b) can become formally equal to Eq. (15a) and this simply by a co-ordinate transformation; in fact on setting: $y = L - x$; $s = L - x'$ ($\Rightarrow \mathrm{d}s = -\mathrm{d}x'$), Eq. (15b) can be written as:

$$\varepsilon^+(y) = C^+ - \lambda \int_y^0 K^+(y - s)\varepsilon^+(s)\,\mathrm{d}s = C^+ + \lambda \int_0^y K^+(y - s)\varepsilon^+(s)\,\mathrm{d}s. \tag{16}$$

Observing also that at $x = y = \frac{L}{2}$, $\varepsilon^-(\frac{L}{2}) = \varepsilon^+(\frac{L}{2})$ the following equality holds true:

$$C^- = C^+ = \frac{C}{2}. \tag{17}$$

It is so sufficient to solve only one of the two Volterra equations (15a) or (15b) to have the complete solution; in fact, denoting with $G(x)$ the solution of (15a) in $[0, x]$, $G(L - x)$ is the solution of (15b) in $[x, L]$, being the complete solution equal to:

$$\varepsilon(x) = G(x) + G(L - x). \tag{18}$$

3.3. Solution of the Volterra equation (15a)

The solution of Eq. (15a) is herein determined for a specific shape of the attenuation function (Eringen, 1987), namely:

$$K(|x-x'|) = \lambda_0 e^{-|x-x'|/\ell} \tag{19}$$

being obviously: $K^-(x-x') = \lambda_0 e^{-(x-x')/\ell}$ and $K^+(x'-x) = \lambda_0 e^{-(x'-x)/\ell}$. Condition (2) yields:

$$\lambda_0 \int_{-\infty}^{\infty} e^{-|x-x'|/\ell}\, dx' = 1 \Rightarrow \lambda_0 = \frac{1}{2\ell} \tag{20a, b}$$

λ_0 denoting a normalization factor. With the above assumption Eq. (15a) writes:

$$\varepsilon^-(x) = C^- + \lambda \int_0^x e^{-(x-x')/\ell} \varepsilon(x')\, dx', \tag{21}$$

in which, on taking into account Eqs. (7), (9) and (20b), $\lambda = -\xi_2/2\ell\xi_1$.

The solution of Eq. (21), obtainable through the method of successive approximations by Neumann's series (see e.g. Krasnov et al., 1983), is:

$$\varepsilon^-(x) = C^- + \lambda C^- \int_0^x \mathscr{R}(x, x', \lambda)\, dx', \tag{22a}$$

where

$$\mathscr{R}(x, x', \lambda) = \sum_{n=1}^{\infty} k_n(x, x') \lambda^{n-1} \tag{22b}$$

is the *resolvent kernel* of the integral equation expressed in the shape of Neumann's series of the *iterated kernels*. The latter are given by:

$$k_1(x, x') \equiv k(x, x'); \quad k_n(x, x') = \int_{x'}^x k(x, \tau) k_{n-1}(\tau, x')\, d\tau. \tag{23}$$

For the assumed kernel (Eq. (19)) it follows:

$$\begin{aligned} k_1(x, x') &= e^{-(x-x')/\ell} \\ k_2(x, x') &= \int_{x'}^x e^{-(x-\tau)/\ell} e^{-(\tau-x')/\ell}\, d\tau = e^{-(x-x')/\ell}(x-x') \\ &\vdots \\ k_n(x, x') &= \int_{x'}^x e^{-(x-\tau)/\ell} e^{-(\tau-x')/\ell} \frac{(\tau-x')^{n-2}}{(n-2)!}\, d\tau = e^{-(x-x')/\ell} \frac{(x-x')^{n-1}}{(n-1)!} \end{aligned} \tag{24}$$

and then:

$$\mathscr{R}(x, x', \lambda) = e^{-(x-x')/\ell} e^{\lambda(x-x')}. \tag{25}$$

On substituting Eq. (25) in (22a), the searched solution is:

$$\varepsilon^-(x) = C^- + \frac{\lambda\ell}{1-\lambda\ell} C^- \left[1 - e^{(\lambda x\ell - x)/\ell}\right]. \tag{26}$$

To determine the constant C^- it is sufficient to observe that at $x = L/2$, $\varepsilon(\frac{L}{2}) = \bar{\varepsilon}$ (refer to Section 3.1), from which, by Eq. (10), the following condition holds true:

$$\varepsilon^-|_{x=\frac{L}{2}} = C^- + \frac{\lambda\ell}{1-\lambda\ell} C^- = \frac{\bar{\varepsilon}}{2} \Rightarrow C^- = \frac{\bar{\varepsilon}}{2}(1-\lambda\ell), \tag{27a, b}$$

where it has been considered that the exponential term in Eq. (26) vanishes for the usual (i.e. physically meaningful) values of ℓ and L. It is easy to verify that, from the above condition, $C_0 = -(\xi_2/2\xi_1)\bar{\varepsilon}$. On combining Eqs. (27b) and (26) the searched solution is finally:

$$\varepsilon^-(x) = \frac{\bar{\varepsilon}}{2}\left[1 - \lambda\ell e^{(\lambda x\ell - x)/\ell}\right], \tag{28}$$

which gives the strain field in $[0, x]$ for the examined problem. The solution ε^+ in $[x, L]$ is given, simply, by setting $x = L - x$ in Eq. (28), i.e.

$$\varepsilon^+(x) = \frac{\bar{\varepsilon}}{2}\left[1 - \lambda\ell e^{(\lambda\ell L - \lambda\ell x - L + x)/\ell}\right] \tag{29}$$

and remembering Eq. (18), the complete solution is:

$$\varepsilon(x) = \bar{\varepsilon} - \frac{\lambda\ell}{2}\bar{\varepsilon}\left[e^{(\lambda x\ell - x)/\ell} + e^{(\lambda\ell L - \lambda\ell x - L + x)/\ell}\right] \quad \forall\, x \in [0, L]. \tag{30}$$

Diagrams of $\varepsilon(x)$ are plotted in the next section for assigned values of $\bar{\varepsilon}$, L and material parameters λ and ℓ.

For completeness the following can be observed: (i) as easily verifiable Eqs. (28) and (29) are the exact solutions of the two Volterra integral equations (15a) and (15b) with constants C and λ previously specified; (ii) in the limit case of $\ell \to 0$, i.e. for a local elastic material, the complete solution (30) gives $\varepsilon(x) \equiv \bar{\varepsilon}$, $\forall x \in [0, L]$ as it has to be; (iii) the theoretical case of a nonlocal bar of infinite length in tension is governed

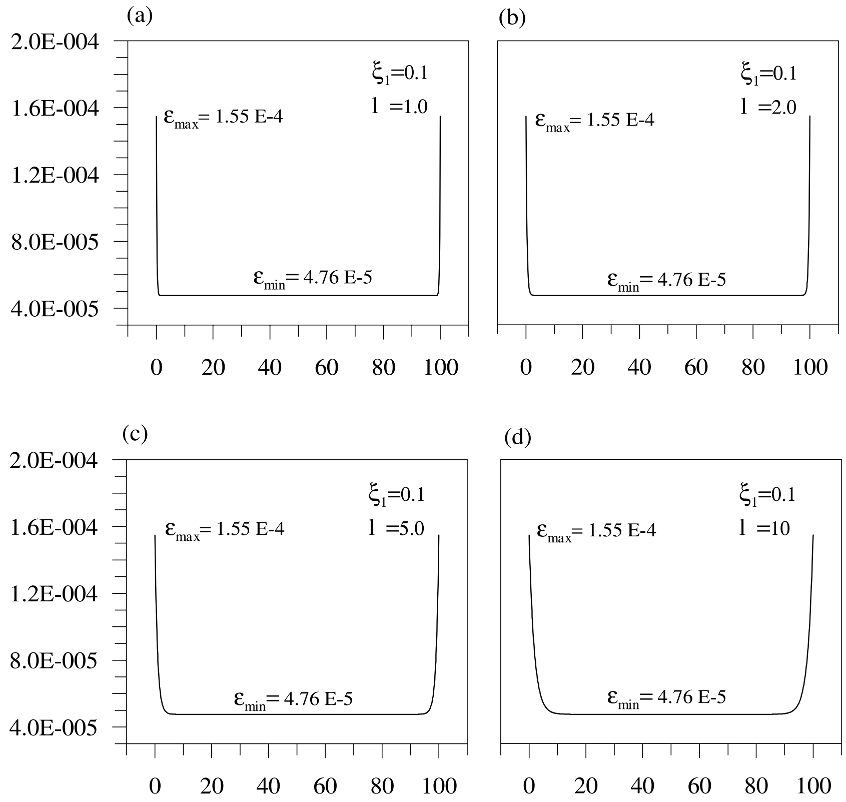

Fig. 4. Exact strain distribution for the bar problem sketched in Fig. 1 assuming $\xi_1 = 0.1$.

by Eq. (7) suitable modified. The latter transforms in an integral equation whose solution, for the assumed kernel (Eq. (19)), can be found in the literature (e.g. Krasnov et al., 1983). As easily verifiable this solution gives $\varepsilon(x) \equiv \bar{\varepsilon}$, $\forall x \in [-\infty, \infty]$ as it has to be.

3.4. Diagrams of the exact solution

The solution to the studied nonlocal bar problem posed in Section 3.1 (refer also to Fig. 1(a)) is hereafter plotted in terms of the strain distributions given by Eq. (30). In Figs. 4 and 5 the strain $\varepsilon(x)$ is reported for different values of the material parameters ξ_1 and ℓ. All the curves have been obtained for $E = 2.1 \times 10^6$ da N/cm^2; $\bar{\sigma} = 100$ da N/cm^2; $A = 1$ cm^2; $L = 100$ cm.

The plotted diagrams all show the typical pattern of strain distribution to be expected for the posed problem. Namely, an increasing trend of the strain values in the bar end portions, with growing strain values towards the end sections, is evidenced. This effect, previously named "boundary effect", is obviously strictly related to the internal length material scale value, ℓ, assumed and it grows for growing ℓ values. To this concern it is worth noting that values of ℓ less than 1 cm have not been reported because the relative results are similar to the ones obtained for $\ell = 1$ cm. Values of ℓ greater than 10 cm are also avoided because they are physically meaningless if compared to the length $L = 100$ cm of the bar specimen. Finally the elastic strain ε coincides with its nonlocal value ($\bar{\varepsilon} = 4.76 \times 10^{-5}$) at all x sufficiently far from the end portions.

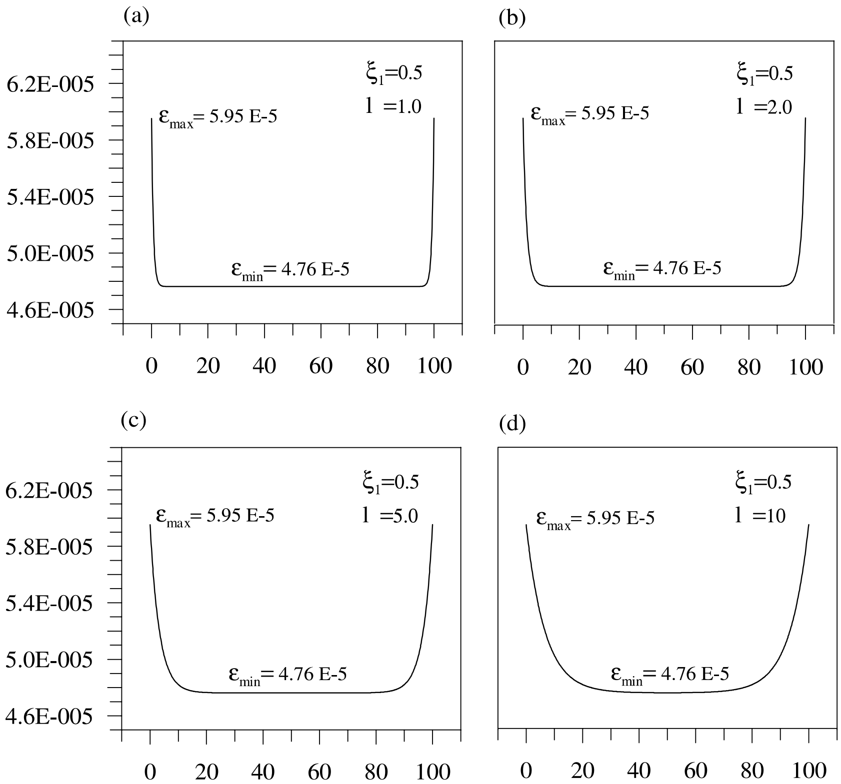

Fig. 5. Exact strain distribution for the bar problem sketched in Fig. 1 assuming $\xi_1 = 0.5$.

4. Conclusion

The exact solution (in terms of strains) for a nonlocal elastic bar of finite length L in (uniform) tension has been determined. The adopted material model is the one known in the literature as the Eringen model for nonlocal (integral) elasticity whose nonlocality features reside only in the constitutive law expressed through a convolution type relation.

The Fredholm integral equation of second kind governing the posed problem has been transformed into two Volterra integral equations of second kind easily solvable by the method of the iterated kernels for the specific choice of the attenuation function.

The proposed resolutive methodology seems to be applicable, at least for the simple mechanical problem here studied, to other analytical forms of the attenuation function entering the constitutive relation as long as they are symmetric.

The exact solution here presented is, in the authors' opinion, an essential result to validate by comparison the effectiveness of approximate and/or numerical procedures aimed to solve mechanical problems whose solution in the context of nonlocal elasticity is not achievable in closed form.

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